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ScienceDirect
Advances in Applied Mathematics 37 (2006) 526–540
ADVANCES IN Applied Mathematics
www.elsevier.com/locate/yaama

On different models of representations of the infinite symmetric group ☆

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Received 3 August 2005; accepted 8 September 2005

Available online 12 May 2006

To Professor Amitai Regev on the occasion of his 65th birthday

Abstract

We present an explicit description of the isomorphism between two models of finite factor representations of the infinite symmetric group: the tableau model in the space of functions on Young bitableaux and the dynamical model in the space of functions on pairs of Bernoulli sequences. The main tool used is the Fourier transform on the symmetric groups. We also start the investigation of the so-called tensor model of two-row representations of the symmetric groups, which plays an intermediate role between the tableau and dynamical models, and show its relations to both these models.

1. Introduction

In this paper, we study several models of irreducible unitary representations and finite factor representations of the infinite symmetric group $\mathfrak{S}_\infty$. The general GNS construction gives the tautological model of unitary representations, which is thus not very suitable for a detailed study. But if the group algebra of the group under consideration has an additional structure, e.g., that of a cross product, as in the case of a locally finite group, then the representation space can also be given a more specific form. Thus, in the case of the infinite symmetric group $\mathfrak{S}_\infty$, considering

☆ Supported by the CRDF grant RUM1-2622-ST-04, RFBR grant 05-01-00899, and INTAS grant 03-51-5018, as well as by the President of Russian Federation grant for Support of Leading Scientific Schools NSh.-2251.2003.1.
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0196-8858/$ – see front matter
doi:10.1016/j.aam.2005.09.007

the Gelfand–Tsetlin algebra and its spectrum, the space of infinite Young tableaux, leads to the so-called "tableau model," see below.

However, for transformation groups, it is most interesting to consider substitutional models of representations, in which the representation space is a space of (vector) functions on a G-space and the group acts by transformations of this G-space. For the finite factor representations of $\mathfrak{S}_\infty$, such a model was found in [7]; we call it the "dynamical" model. Its construction is based on the so-called trajectory (or groupoid) realization of dynamical systems. The most unexpected feature of this construction is that for nondegenerate values of the Thoma parameters, the representation turns out to be "purely substitutional," i.e., the algebra generated by the group representation operators only (without the diagonal subalgebra) gives an irreducible representation of the groupoid algebra. The reason is that the group action is nonfree; see Section 7 for a discussion of nonfree group actions.

The main result of this paper is an explicit description of the isomorphism between the tableau and dynamical models; its construction uses the Fourier theory for the symmetric groups (see [10]). Each of the models under consideration has its own maximal commutative subalgebra, and the relation between them is quite complicated. The intertwining operator acting from the tableau model to the dynamical one does preserve the subspaces of n-cylinder functions (i.e., can be defined on finite approximations), while the conjugate operator acting in the opposite direction does not.

Substitutional models of representations of the finite symmetric groups are not sufficiently studied and, apparently, hide untapped opportunities. In this paper, we also consider the so-called "tensor" model of two-row representations of the symmetric groups, which was suggested by the second author and investigated in [3]. In a sense, it plays the role of a "bridge" between the tableau and dynamical models of factor representations of $\mathfrak{S}_\infty$ and yields an interpretation of another two representations arising in these models—the "diagonal" representation in the dynamical model and the "concomitant" representation in the tableau model. A more detailed study of the tensor model will be presented elsewhere.

The paper is organized as follows. In Section 2, we recall necessary facts and introduce notation related to the representation theory of the finite and infinite symmetric groups. In Sections 3 and 4, we describe the tableau and dynamical models of finite factor representations of $\mathfrak{S}_\infty$, respectively. Section 5 contains the main result of the paper, an explicit description of the isomorphism between the tableau and dynamical models. In Section 6, we describe the tensor model of two-row representations and its relations to the tableau and dynamical models. Finally, Section 7 contains a discussion of nonfree group actions, which is closely related to the dynamical model.

2. Necessary background and notation

In this section, we recall necessary facts and introduce notation related to the representation theory of the symmetric groups. For the basic notions of the representation theory of the finite symmetric groups, see, e.g., [1,2]. Necessary facts from the representation theory of the infinite symmetric group can be found, e.g., in [9]. The Fourier theory for the finite and infinite symmetric groups is described in [10].

2.1. Finite symmetric groups

We denote by $\mathfrak{S}_n$ the symmetric group of degree n and by $\mathbb{C}[\mathfrak{S}_n]$ the group algebra of $\mathfrak{S}_n$.

The irreducible representations of the symmetric group $\mathfrak{S}_n$ are indexed by the set $\mathbb{Y}_n$ of Young diagrams with n cells. Let π_λ be the irreducible unitary representation of $\mathfrak{S}_n$ corresponding to a diagram $\lambda \in \mathbb{Y}_n$, and let χ_λ and $\dim\lambda$ be the character and the dimension of π_λ, respectively.

The branching of the irreducible representations of the symmetric groups is described by the Young graph $\mathbb{Y}$. The vertex set of the graded graph $\mathbb{Y}$ is $\bigcup_n \mathbb{Y}_n$, and two vertices $\mu \in \mathbb{Y}_{n-1}$ and $\lambda \in \mathbb{Y}_n$ are joined by an edge if and only if $\mu \subset \lambda$. By definition, the zero level $\mathbb{Y}_0$ consists of the empty diagram $\emptyset$.

Denote by $T(\lambda)$ the set (consisting of $\dim\lambda$ elements) of Young tableaux of shape $\lambda \in \mathbb{Y}_n$, or, which is the same, the set of paths in the Young graph from the empty diagram $\emptyset$ to λ. Let $\{h_t, t \in T(\lambda)\}$ be the Gelfand–Tsetlin basis in the space V_λ of the representation π_λ, and denote by $\operatorname{End} V_\lambda$ the algebra of matrices in the space V_λ. Let $T_n = \bigcup_{\lambda\in\mathbb{Y}_n} T(\lambda)$ be the set of Young tableaux with n cells.

Denote by $B_n = \{(s,t)\colon\ s,t \in T_n \text{ are of the same shape}\}$ the set of bitableaux of size n (i.e., the set of pairs of paths of length n in the Young graph ending at the same vertex).

The (matrix form of the) Fourier transform on the symmetric group $\mathfrak{S}_n$ associates with a function $f \in \mathbb{C}[\mathfrak{S}_n]$ a matrix-valued function $\hat f$ on $\mathbb{Y}_n$, where $\hat f(\lambda) \in \operatorname{End} V_\lambda$ is given by the formula

$$\hat f(\lambda) = \sum_{w\in\mathfrak{S}_n} f(w)\pi_\lambda(w).$$

Considering the matrix elements with respect to the Gelfand–Tsetlin basis, we obtain the tableau form of the Fourier transform: $f \mapsto \hat f \in \mathbb{C}(B_n)$, where

$$\hat f(s,t) = \sum_{w\in\mathfrak{S}_n} f(w)\bigl(\pi_\lambda(w)h_s, h_t\bigr), \quad (s,t) \in B_n.$$

The inversion formula for the Fourier transform reads as

$$f(w) = \sum_{\lambda\in\mathbb{Y}_n} \frac{\dim\lambda}{n!} \operatorname{tr}\bigl(\hat f(\lambda)\pi^*_\lambda(w)\bigr), \quad w \in \mathfrak{S}_n \tag{1}$$

(here $*$ stands for the matrix conjugation). Under the Fourier transform, convolution of functions on $\mathfrak{S}_n$ goes to matrix multiplication: $\widehat{(f*g)}(\lambda) = \hat f(\lambda)\hat g(\lambda)$.

2.2. *Infinite symmetric group*

Now let $\mathfrak{S}_\infty = \bigcup_{n=1}^\infty \mathfrak{S}_n = \varinjlim \mathfrak{S}_n$ be the infinite symmetric group with the fixed structure of an inductive limit of finite groups, and let $\mathbb{C}[\mathfrak{S}_\infty]$ be the group algebra of $\mathfrak{S}_\infty$.

Denote by $T = \varprojlim T_n$ the space of infinite Young tableaux (the projective limit of T_n with respect to the natural projections forgetting the tail of a path). With the topology of coordinatewise convergence T is a totally disconnected metrizable compact space. The tail equivalence relation $\sim$ on T is defined as follows: for paths $s = (s_1, s_2, \ldots)$ and $t = (t_1, t_2, \ldots)$, we have $s \sim t$ if and only if $s_k = t_k$ for sufficiently large k. The n-equivalence relation $\sim_n$ on T is defined in a similar way: $s \sim_n t$ if and only if $s_k = t_k$ for $k \geqslant n$. Denote by $[t]_n \in T_n$ the initial segment of length n of a tableau $t \in T$.

Consider the space $\mathcal{B} = \{(s,t)\colon\ s,t \in T, s \sim t\}$ of infinite bitableaux with the inductive limit topology $\mathcal{B} = \varinjlim \mathcal{B}_n$, where $\mathcal{B}_n = \{(s,t)\colon\ s,t \in T, s \sim_n t\}$. Thus $\mathcal{B}$ is a separable totally disconnected locally compact space, and its diagonal $\mathcal{B}_0 = \{(t,t), t \in T\}$ is homeomorphic to T. Note that the space of bitableaux can be regarded as the principal groupoid generated by the tail equivalence relation on T (see [5]). The unit space of this groupoid can be identified with the diagonal $\mathcal{B}_0 \sim T$, and for an arbitrary tableau $t \in T$, its preimage $G^t = r^{-1}(t)$ under the range map r is the countable set of bitableaux $G^t = \{(t,\cdot) \in \mathcal{B}\}$ with the first component equal to t. Denote by λ^t the counting measure on G^t.

The Fourier transform establishes a canonical isomorphism between the group algebra $\mathbb{C}[\mathfrak{S}_\infty]$ of the infinite symmetric group and the $*$-algebra $\mathbb{C}(\mathcal{B})$ of locally constant finitary functions on the space of bitableaux $\mathcal{B}$ with the multiplication

$$fg(s,t) = \sum_{r \sim t} f(s,r)g(r,t) \tag{2}$$

and involution

$$f^*(s,t) = \overline{f(t,s)}.$$

This is just the realization of $\mathbb{C}[\mathfrak{S}_\infty]$ as the cross product constructed from the commutative algebra of functions on the space of tableaux T (Gelfand–Tsetlin algebra) and the tail equivalence relation [6,8,9]. Here the cross product, as a C^*-algebra, is the groupoid C^*-algebra generated by the space of all Young tableaux and the tail equivalence relation. We can also choose one transformation (the so-called Young-adic shift) that together with the Gelfand–Tsetlin algebra generates the whole cross product.

A measure M on T is called central if the measure $M(\{t\colon\ [t]_n = s\})$ of a cylinder set depends only on the shape of the tableau $s \in T_n$. The Fourier transform on the infinite symmetric group (see [10]) determines, in particular, a correspondence $\chi \leftrightarrow M$ between the characters of the infinite symmetric group and the central probability measures on T; it is given by the formula

$$\chi_n = \sum_{\lambda \in \mathbb{Y}_n} M_n(\lambda) \frac{\chi^\lambda}{\dim \lambda}, \tag{3}$$

where $\chi_n = \chi|_{\mathfrak{S}_n}$ is the restriction of a character χ to $\mathfrak{S}_n$, and $M_n(\lambda) = M(\{t\colon\ [t]_n \in \lambda\})$ is the cylinder distribution on $\mathbb{Y}_n$ of the measure M.

It is well known (see, e.g., [9]) that the finite factor representations of the infinite symmetric group $\mathfrak{S}_\infty$ are indexed by two nonincreasing sequences of nonnegative real numbers (Thoma parameters) $\alpha = \{\alpha_i, i = 1,2,\ldots\}$ and $\beta = \{\beta_i, i = 1,2,\ldots\}$, where

$$\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant 0, \quad \beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant 0, \quad \sum \alpha_i + \sum \beta_i \leqslant 1.$$

Denote the representation corresponding to a pair (α,β) by $\pi^{\alpha,\beta}$ and its character by $\phi^{\alpha,\beta}$.

There are different models of finite factor representations of the infinite symmetric group. The most important of them are the "tableau model" obtained by the GNS construction and the "dynamical model" suggested in [7].

3. Tableau model of the factor representations

The standard GNS construction yields the so-called tableau model of finite factor representations of the infinite symmetric group $\mathfrak{S}_\infty$.

Let $M^{\alpha,\beta}$ be the central measure on T corresponding to the character $\phi^{\alpha,\beta}$ according to (3), and let $\{M_n^{\alpha,\beta}\}$ be the family of cylinder distributions of the measure $M^{\alpha,\beta}$. Then

$$\sum_{\lambda\in\mathbb{Y}_n} M_n^{\alpha,\beta}(\lambda)\frac{\chi^\lambda(\sigma)}{\dim\lambda}=\phi^{\alpha,\beta}(\sigma),\quad \sigma\in\mathfrak{S}_n.$$

Consider the measure $\widetilde{M}^{\alpha,\beta}$ on the groupoid $\mathcal{B}$ induced by the measure $M^{\alpha,\beta}$ on its diagonal T:

$$\widetilde{M}^{\alpha,\beta}=\int_T \lambda^t\, dM^{\alpha,\beta}(t).$$

Denote by $H^{\alpha,\beta}=L^2(\mathcal{B},\widetilde{M}^{\alpha,\beta})$ the space of square integrable functions on $\mathcal{B}$ with respect to this measure. Thus $f\in H^{\alpha,\beta}$ if and only if

$$\int_T \sum_{s\sim t}\big|f(t,s)\big|^2\, dM^{\alpha,\beta}(t)<\infty,$$

and the scalar product in $H^{\alpha,\beta}$ is given by

$$(f,g)=\int_T\sum_{s\sim t} f(t,s)\overline{g(t,s)}\, dM^{\alpha,\beta}(t).$$

If f and g are n-cylinder functions, i.e., are supported by the set $\mathcal{B}_n$ of pairs of n-equivalent paths and depend only on the initial n-segments of paths (so that they can essentially be regarded as functions on B_n), then

$$(f,g)=\sum_{\lambda\in\mathbb{Y}_n}\frac{M_n^{\alpha,\beta}(\lambda)}{\dim\lambda}\operatorname{tr}\big(\hat f(\lambda)\hat g^*(\lambda)\big).$$

The representation $U\approx\pi^{\alpha,\beta}$ of $\mathfrak{S}_\infty$ in $H^{\alpha,\beta}$ is defined by the following formula. Given $g\in\mathfrak{S}_n$, let $\hat\delta_g$ be the Fourier transform of the δ-function at g given by the formula

$$\hat\delta_g(s,t)=\begin{cases}(\pi_\lambda(g)h_{[s]_n},h_{[t]_n}) & \text{if } s\sim_n t,\ [s]_n,[t]_n\in\lambda,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$U_g f=\hat\delta_g f,\quad g\in\mathfrak{S}_\infty,$$

where the multiplication is understood in the sense (2). For n-cylinder functions f, we have

$$(U_g f)(\lambda)=\pi_\lambda(g)f(\lambda),\quad g\in\mathfrak{S}_n,\ \lambda\in\mathbb{Y}_n.$$

The characteristic function of the diagonal

$$\Psi(s,t)=\begin{cases}1 & \text{if } s=t,\\ 0 & \text{otherwise}\end{cases}$$

is a cyclic vector for this representation, and the character $\phi^{\alpha,\beta}$ is the corresponding spherical function:

$$(U_g\Psi,\Psi)=\phi^{\alpha,\beta}(g).$$

4. Dynamical model of the factor representations

The dynamical model of finite factor representations of the infinite symmetric group was suggested in [7]; see also [9] and a modification in [4].

Let us consider the simplest case where $\sum\alpha_i=1$ and $\beta_i=0$, $i=1,2,\dots$. Then the sequence α can be regarded as a measure on the set $\mathbb{N}$ of positive integers.

Consider the space of sequences $\mathcal{X}=\prod_{k=1}^{\infty}\mathbb{N}$ with the product measure $m_\alpha=\prod_{k=1}^{\infty}\alpha$. The infinite symmetric group $\mathfrak{S}_\infty$ acts on $\mathcal{X}$ by substitutions of coordinates, and this action preserves the measure m_α. Define an equivalence relation $\sim$ on $\mathcal{X}$ as follows: $x\sim y$ if there exists $\sigma\in\mathfrak{S}_\infty$ such that $y=\sigma x$. Let $\widetilde{\mathcal{X}}=\{(x,y)\colon\ x,y\in\mathcal{X}, x\sim y\}$ be the principal groupoid with diagonal $\mathcal{X}$ constructed from this equivalence relation. Consider the measure $\widetilde{m}_\alpha$ on the groupoid $\widetilde{\mathcal{X}}$ induced by the measure m_α on the diagonal $\mathcal{X}$, and set $\mathcal{K}_\alpha=L^2(\mathcal{X},\widetilde{m}_\alpha)$. Thus

$$\mathcal{K}_\alpha=\left\{h:\widetilde{\mathcal{X}}\to\mathbb{C}\colon\ \|h\|^2=\int\limits_{\mathcal{X}}\sum_{y\sim x}\bigl|h(x,y)\bigr|^2\,dm_\alpha(x)<\infty\right\},$$

and the scalar product in $\mathcal{K}_\alpha$ is given by

$$(h_1,h_2)=\int\limits_{\mathcal{X}}\sum_{y\sim x}h_1(x,y)\overline{h_2(x,y)}\,dm_\alpha(x).$$

The representation $V\approx\pi^\alpha=\pi^{\alpha,0}$ of the infinite symmetric group $\mathfrak{S}_\infty$ in the space $\mathcal{K}_\alpha$ is given by the formula

$$(V_gh)(x,y)=h\bigl(g^{-1}x,y\bigr). \tag{4}$$

Let $\Phi\in\mathcal{K}_\alpha$ be the characteristic function of the diagonal:

$$\Phi(x,y)=\begin{cases}1 & \text{if } x=y,\\ 0 & \text{otherwise.}\end{cases}$$

We will regard the representation V in the cyclic hull of the vector Φ (it coincides with the whole space $\mathcal{K}_\alpha$ if the sequence α consists of pairwise distinct elements). We have

$$(V_g\Phi,\Phi)=\phi_\alpha(g)=m_\alpha\bigl(\{x\in\mathcal{X}\colon\ gx=x\}\bigr)=\prod_{k\geqslant 2}\left(\sum_i\alpha_i^k\right)^{r_k(g)},$$

where $r_k(g)$ is the number of cycles of length k in a permutation $g\in\mathfrak{S}_\infty$.

5. Isomorphism between the tableau model and the dynamical model

Denote by H_n^α the subspace in $H^\alpha = H^{\alpha,0}$ that consists of n-cylinder functions.

Theorem 1. *An isomorphism of the tableau model and the dynamical model of the factor representation $\pi^\alpha = \pi^{\alpha,0}$ is given by the following intertwining operator. Let $f \in H^\alpha = L^2(\mathcal{B}, M^\alpha)$. Denote by f_n the projection of f to H_n^α. Then*

$$Tf = \lim_{n\to\infty} Tf_n, \qquad (Tf_n)(x,y) = \sum_{\substack{\sigma\in\mathfrak{S}_n,\\ x=\sigma y}} \hat{f}_n^{-1}(\sigma), \quad (x,y) \in \widetilde{\mathcal{X}}, \tag{5}$$

where $\hat{f}_n^{-1}$ is the inverse Fourier transform, given by formula (1), *of the n-cylinder function f_n.*

The conjugate operator $S = T^*$ can be described as follows. Let $g \in \mathcal{K}_\alpha$. Consider the function G on $\mathfrak{S}_\infty$ given by the formula

$$G(w) = \int\limits_{\mathcal{X}} g(wx, x)\, dm_\alpha(x), \quad w \in \mathfrak{S}_\infty, \tag{6}$$

and denote by G_n its restriction to $\mathfrak{S}_n$. Then

$$Sg = \lim_{n\to\infty} S_n g,$$

where

$$(S_n g)(\lambda) = \frac{1}{\widehat{\phi_n^\alpha}(\lambda)} \widehat{G}_n(\lambda) = \frac{\dim^2\lambda}{n! M_n^\alpha(\lambda)} \widehat{G}_n(\lambda);$$

here ϕ_n^α is the restriction of the character ϕ^α to $\mathfrak{S}_n$.

Remark 1. Note that the operator T sends cylinder functions on $\mathcal{B}$ (i.e., functions depending on the initial n-segments of paths) to cylinder functions on $\widetilde{\mathcal{X}}$ (i.e., functions depending on the first n coordinates of sequences), while the conjugate operator S does not satisfy this property.

Remark 2. The theorem can be generalized to the case of an arbitrary finite factor representation (when β is not necessarily zero).

Proof. Using the Fourier transform, we obtain a realization of the tableau model on $\mathfrak{S}_n$ in the group algebra $\mathbb{C}[\mathfrak{S}_n]$ with the scalar product

$$(a,b) = \sum_{u,v\in\mathfrak{S}_n} a(u)\overline{b(v)}\phi_\alpha\big(v^{-1}u\big), \quad a, b \in \mathbb{C}[\mathfrak{S}_n],$$

and the natural group action:

$$(U_g a)(w) = a\big(g^{-1}w\big).$$

The cyclic vector corresponding to Ψ is the δ-function at the identity element δ_e.

Consider a δ-function $\delta_g \in \mathbb{C}[\mathfrak{S}_n]$, $g \in \mathfrak{S}_n$. Obviously, $\delta_g = U_g\delta_e$. Since $T\Psi = \Phi$, we have $T(U_g\Psi) = V_g\Phi$, so that δ_g goes to the function

$$(T\delta_g)(x,y) = (V_g\Phi)(x,y) = \Phi\bigl(g^{-1}x, y\bigr) = \begin{cases} 1 & \text{if } x = gy, \\ 0 & \text{otherwise.} \end{cases}$$

By linearity, for $f \in H_n^\alpha$,

$$(Tf)(x,y) = \sum_{w\in\mathfrak{S}_n} \hat{f}^{-1}(w)\Phi\bigl(w^{-1}x, y\bigr) = \sum_{\substack{w\in\mathfrak{S}_n,\\ x=wy}} \hat{f}^{-1}(w),$$

as required.

In order to prove the formula for the conjugate operator, consider functions $f \in H_n^\alpha$ and $g \in \mathcal{K}_\alpha$ and the scalar product

$$\begin{aligned}(Tf,g) &= \int_{\mathcal{X}} (Tf)(x,y)\sum_{y\sim x}\overline{g(x,y)}\,dm_\alpha(x) = \int_{\mathcal{X}} \sum_{\substack{\sigma\in\mathfrak{S}_n\\ x=\sigma y}} \hat{f}^{-1}(\sigma)\sum_{y\sim x}\overline{g(x,y)}\,dm_\alpha(x) \\ &= \sum_{\sigma\in\mathfrak{S}_n}\hat{f}^{-1}(\sigma)\int_{\mathcal{X}}\overline{g(\sigma x, x)}\,dm_\alpha(x) = \sum_{\sigma\in\mathfrak{S}_n}\hat{f}^{-1}(\sigma)\overline{G_n(\sigma)}.\end{aligned}$$

Applying the inversion formula (1) for the Fourier transform, we obtain

$$\begin{aligned}(Tf,g) &= \sum_{\sigma\in\mathfrak{S}_n}\sum_{\lambda\in\mathbb{Y}_n}\frac{\dim\lambda}{n!}\operatorname{tr}\bigl(f(\lambda)T_\lambda^*(\sigma)\bigr)\overline{G_n(\sigma)} \\ &= \sum_{\lambda\in\mathbb{Y}_n}\frac{\dim\lambda}{n!}\operatorname{tr}\Bigl(f(\lambda)\sum_{\sigma\in\mathfrak{S}_n}\overline{G_n(\sigma)}T_\lambda^*(\sigma)\Bigr) = \sum_{\lambda\in\mathbb{Y}_n}\frac{\dim\lambda}{n!}\operatorname{tr}\bigl(f(\lambda)\widehat{G}_n^*(\lambda)\bigr).\end{aligned}$$

Comparing with the formula

$$(Tf,g) = (f,Sg) = \sum_{\lambda\in\mathbb{Y}_n}\frac{M_n^\alpha(\lambda)}{\dim\lambda}\operatorname{tr}\bigl(f(\lambda)(Sg)^*(\lambda)\bigr)$$

yields the desired result. $\square$

Example. Let $g = \Phi$ be the characteristic function of the diagonal. Then

$$G(w) = \int_{\mathcal{X}}\Phi(wx,x)\,dm_\alpha(x) = m_\alpha\bigl(\{x\in\mathcal{X}\colon\ wx = x\}\bigr) = \phi^\alpha(w),$$

so that $\widehat{G}_n(\lambda) = \sum_{\mu\in\mathbb{Y}_n}\frac{M_n^\alpha(\mu)}{\dim\mu}\widehat{\chi}^\mu(\lambda) = \frac{M_n^\alpha(\lambda)}{\dim\lambda}\frac{n!}{\dim\lambda}E_\lambda$, whence

$$(Sg)(\lambda) = \frac{\dim^2\lambda}{n!M_n^\alpha(\lambda)}\widehat{G}_n(\lambda) = E_\lambda,$$

i.e., $Sg = \Psi$.

Corollary 1. *A function $g \in \mathcal{K}_\alpha$ is the image of a central function if and only if*

$$g(x, y) = g(wx, wy) \quad \textit{for all } w \in \mathfrak{S}_\infty.$$

6. The tensor model of two-row representations and the concomitant representation

In this section, we describe the so-called tensor model of irreducible representations of the symmetric groups corresponding to two-row diagrams, which was suggested by the second author and investigated in [3]. In particular, using this model, one can construct the so-called concomitant representation, an irreducible representation of $\mathfrak{S}_\infty$ associated in a natural way with its factor representation.

6.1. Finite case

For $0 \leqslant k \leqslant n$, denote by $F_{n,k}$ the set of k-element subsets of the set $\{1, \ldots, n\}$. Given $I = \{i_1, \ldots, i_k\} \in F_{n,k}$, let $x_I = x_{i_1} \cdots x_{i_k}$.

Let $A_{n,k} = \{\sum_{I \in F_{n,k}} c_I x_I\}$ be the vector space of square-free forms of degree k in n variables. This space can also be identified with the space of symmetric tensors of rank k with zero diagonal components over the n-dimensional space (a function $f = \sum_{I \in F_{n,k}} c_I x_I$ is identified with the tensor $\{T_{j_1,\ldots,j_k}\}_{j_1,\ldots,j_k=1}^n$, where $T_{j_1,\ldots,j_k} = c_{\{j_1,\ldots,j_k\}}$ if the indices $j_1, \ldots, j_k$ are pairwise distinct and $T_{j_1,\ldots,j_k} = 0$ otherwise). In what follows, we will use this identification without explicitly mentioning.

For $f = \sum_{I \in F_{n,k}} c_I x_I \in A_{n,k}$, let $\|f\|^2 = \sum_{I \in F_{n,k}} |c_I|^2$.

Let $A_{n,k}^0$ be the subspace of $A_{n,k}$ determined by the "zero conditions" of the form "the sum along each one-dimensional direction vanishes":

$$A_{n,k}^0 = \left\{ \sum c_I x_I \in A_{n,k} \;\middle|\; \sum_{j \notin I'} c_{I' \cup j} = 0 \text{ for every } I' \in F_{n,k-1} \right\}.$$

There is a natural action of the symmetric group $\mathfrak{S}_n$ on the space $A_{n,k}$ by substitutions of indices: given $\sigma \in \mathfrak{S}_n$,

$$\sigma \cdot \sum_{I \in F_{n,k}} c_I x_I = \sum_{I \in F_{n,k}} c_I x_{\sigma I}, \quad \text{where } \sigma\{i_1, \ldots, i_k\} = \big\{\sigma(i_1), \ldots, \sigma(i_k)\big\},$$

or, in tensor form, $\sigma\{T_{j_1,\ldots,j_k}\} = \{T'_{j_1,\ldots,j_k}\}$, where $T'_{j_1,\ldots,j_k} = T_{\sigma^{-1}(j_1),\ldots,\sigma^{-1}(j_k)}$. It is easy to see that the subspace $A_{n,k}^0$ is invariant under this action. Note that the spaces $A_{n,k}^0$ and $A_{n,n-k}^0$, as well as the corresponding representations of $\mathfrak{S}_n$, are obviously equivalent.

Theorem 2. [3] (1) *Let $k \leqslant n/2$. The representation of the symmetric group $\mathfrak{S}_n$ in the space $A_{n,k}^0$ (and in the space $A_{n,n-k}^0$) is equivalent to the irreducible representation $\pi_{n-k,k}$ corresponding to the two-row diagram $\lambda_{n,k} = (n-k, k)$ with rows of lengths $n-k$ and k.*

(2) *The representation of the symmetric group $\mathfrak{S}_n$ in the space $A_{n,k}$ is equivalent to the direct sum of $\pi_{n-l,l}$ over all $l = 0, 1, \ldots, k$. In particular, the representation of $\mathfrak{S}_n$ in $A_{n,[n/2]}$ is equivalent to the multiplicity-free direct sum*

$$\bigoplus_{k=0}^{[n/2]} \pi_{n-k,k}$$

of all two-row representations.

These realizations of representations of $\mathfrak{S}_n$ in spaces of tensors (or, equivalently, square-free forms) will be called *tensor realizations*.

Remark. Thus the representation of $\mathfrak{S}_n$ in the space $A_{n,[n/2]}$ of all symmetric tensors of rank $[n/2]$ with zero diagonal components over the n-dimensional space is the *two-row model representation* in the sense of Gelfand: it contains each representation corresponding to a two-row diagram exactly once.

Denote by $H_{n,k}$ the subspace in $A_{n,[n/2]}$ corresponding to the representation $\pi_{n-k,k}$. Thus

$$A_{n,[n/2]} = \bigoplus_{k=0}^{[n/2]} H_{n,k}, \tag{7}$$

and $H_{n,k}$ is isomorphic to $A^0_{n,k}$.

On the other hand, the two-row model representation can be realized in the space of all two-row tableaux with the standard action of $\mathfrak{S}_n$. So the problem is to establish an isomorphism between these two realizations.

The difficulty is that the embeddings $A_{n,[n/2]} \mapsto A_{n+1,[n+1/2]}$ determined by the structure of tableaux, i.e., by the branching of irreducible representations of the symmetric groups, are nontrivial. For example (see the general formula below), the embedding of the space $A^0_{n,1}$, which can be interpreted as the space $\{(b_1, \ldots, b_n) \mid \sum b_i = 0\}$ of n-vectors summing to zero, to the space $A^0_{n+1,2} = \{(a_{i,j})_{i,j=1}^{n+1} \mid a_{i,j} = a_{j,i}, a_{i,i} = 0, \sum_i a_{i,j} = \sum_j a_{i,j} = 0\}$ of symmetric $(n+1) \times (n+1)$ matrices satisfying the appropriate zero conditions is given by the following formula: for $i < j$,

$$a_{i,j} = a_{j,i} = \begin{cases} b_i + b_j & \text{if } j \leqslant n; \\ -(n-2)b_i & \text{if } j = n+1. \end{cases}$$

The choice of a central two-row Thoma measure determines the corresponding norms in the spaces $A_{n,k}$. Namely, given $p \in (0, \frac{1}{2}]$, consider the sequence $\alpha = (1-p, p, 0, 0, \ldots)$. Let M^α be the corresponding central Thoma measure on the space T of infinite Young tableaux (see Section 3). Note that M^α is supported by two-row tableaux. Let us introduce a Hilbert norm $\|\cdot\|_{\alpha,n}$ in $A_{n,[n/2]}$ as follows. Given $f \in A_{n,[n/2]}$, consider the decomposition of f according to (7): $f = \sum f_k$, where $f_k \in H_{n,k}$, and let

$$\|f\|^2_{\alpha,n} = \sum_{k=0}^{[n/2]} \frac{M^\alpha_n(\lambda_{n,k})}{\dim \lambda_{n,k}} \|f_k\|^2,$$

where M_n^α is the cylinder distribution of the measure M^α on $\mathbb{Y}_n$. We will call $\|\cdot\|_{\alpha,n}$ the α-norm in $A_{n,[n/2]}$. Note that the action of $\mathfrak{S}_n$ in $A_{n,[n/2]}$ is unitary with respect to this norm.

Now let us turn to the infinite case. We will consider two different inductive limits of tensor representations determined by different embeddings and leading to the so-called *diagonal* (*Fock*) and *concomitant* representations of $\mathfrak{S}_\infty$.

6.2. Diagonal representation

The first inductive limit gives a tensor realization of the *diagonal representation* of $\mathfrak{S}_\infty$ in the dynamical model. Namely, let m_α be the Bernoulli measure on the space $\mathcal{X}$ (which in this case is the space $\{0,1\}^{\mathbb{N}}$ of 0–1 sequences) corresponding to the Thoma parameter α (see Section 4). Recall that there is a one-to-one correspondence between finite factor representations of $\mathfrak{S}_\infty$ and spherical representations of the Gelfand pair $(\mathfrak{S}_\infty \times \mathfrak{S}_\infty, \operatorname{diag}\mathfrak{S}_\infty)$. The representation of $\mathfrak{S}_\infty \times \mathfrak{S}_\infty$ corresponding to π^α in the dynamical model is given by the formula $(V_{(g_1,g_2)}h)(x,y) = h(g_1^{-1}x, g_2^{-1}y)$ (cf. (4)). The subspace of functions supported by the diagonal is obviously invariant with respect to the action of the diagonal subgroup $\operatorname{diag}\mathfrak{S}_\infty = \{(g,g) \mid g \in \mathfrak{S}_\infty\}$ (isomorphic to $\mathfrak{S}_\infty$) of $\mathfrak{S}_\infty \times \mathfrak{S}_\infty$. The obtained representation of $\mathfrak{S}_\infty$ in $L^2(\mathcal{X}, m_\alpha)$ is called the *diagonal representation*. Of course, it is just the natural action of $\mathfrak{S}_\infty$ by substitutions: $\sigma f(x) = f(\sigma^{-1}x)$. Note that this representation can be extended to a representation of the group $\mathfrak{S}^\infty$ of *all* permutations of $\mathbb{N}$.

Now consider the space $A_n = \bigoplus_{k=0}^n A_{n,k}$ of all square-free forms in n variables. Obviously, it can be identified with the space of functions on the set $\{0,1\}^n$ of finite 0–1 sequences (since each form from A_n is uniquely determined by its restriction to the unit cube). Since the space $A_{n,k}$ can be regarded as a subspace of $A_{n+1,k}$, we can consider the identity embedding $A_n \mapsto A_{n+1}$. It agrees with the natural embedding of the space of functions on n-sequences to the space of functions on $(n+1)$-sequences and commutes with the action of the group $\mathfrak{S}_n$. Now consider the Bernoulli measure $m_{\alpha,n} = \prod_{k=1}^n \alpha$ on $\{0,1\}^n$ and denote by $|\cdot|_{\alpha,n}$ the image in A_n of the corresponding L^2 norm under the above identification (it is easy to write an explicit formula for this norm, but we do not need it). We obtain the following proposition.

Proposition 1. *The inductive limit $\mathfrak{A}$ of the Hilbert spaces $(A_n, |\cdot|_{\alpha,n})$ with respect to the identity embeddings is isometric to the space $L^2(\mathcal{X}, m_\alpha)$, and the corresponding inductive limit τ_α of tensor representations is equivalent to the diagonal representation of $\mathfrak{S}_\infty$ in $L^2(\mathcal{X}, m_\alpha)$.*

Remark. For all $\alpha = (1-p, p, 0, \ldots)$, $p \in (0, 1/2]$, the diagonal representations are equivalent, namely, each of them is the (multiplicity-free) direct sum of irreducible representations of $\mathfrak{S}_\infty$ over all infinite two-row Young diagrams with finite second row (each such representation is the inductive limit of the irreducible representations of $\mathfrak{S}_n$ corresponding to diagrams with fixed second row and growing first row).

For $\alpha = 1/2$, the space $\mathfrak{A}$ is isometric to the canonical fermion Fock space, and the decomposition of τ_α into irreducible representations of $\mathfrak{S}_\infty$ coincides with the multi-particle decomposition of the fermion Fock space.

6.3. Concomitant representation

Recall (see Section 2.2) that the group algebra of $\mathfrak{S}_\infty$ is the cross product (or groupoid algebra) constructed from the commutative algebra of functions on the space of tableaux T and the tail equivalence relation. According to the general theory, given a finite measure M on T invariant with respect to the equivalence relation, we can construct a type II_1 factor representation (von Neumann representation) of the groupoid algebra and also an irreducible representation (Koopman representation), acting in the space $L^2(T, M)$, of the same algebra. In our case, the von Neumann representation is just the tableau model considered in Section 3. Now we are going to consider the corresponding Koopman representation.

Definition 1. Let M be a central measure on the space T of infinite Young tableaux. The Koopman[1] representation in the space $L^2(T, M)$ of complex-valued functions on the space of tableaux is called the concomitant representation of the infinite symmetric group $\mathfrak{S}_\infty$ associated with the measure M.

An explicit description of the concomitant representation of $\mathfrak{S}_\infty$ is as follows. Observe that if a function $f \in \mathbb{C}(B_n)$ does not depend on the second component, then the function $U_g f = \hat{\hat{\delta}}_g f$ also satisfies this property. Thus the subspace of such functions (which can be identified with the space of functions on T_n) is invariant under U, so that we obtain a unitary representation $\widetilde{U}_g$ of $\mathfrak{S}_n$ in the space $L^2(T_n, M_n)$, where M_n is the cylinder distribution of the measure M on T_n. Moreover, these representations form an inductive family with respect to the natural embeddings $L^2(T_n, M_n) \mapsto L^2(T_{n+1}, M_{n+1})$. The concomitant representation is just the inductive limit of these representations.

Now we will show that the concomitant representation associated with a central measure concentrated on two-row tableaux can be constructed as an inductive limit of tensor representations.

Consider another embedding $i_n : A_{n,[n/2]} \to A_{n+1,[n+1/2]}$, which is determined by the branching of irreducible representations of the symmetric groups, i.e., by the structure of the Young graph (more exactly, by its restriction to the set of two-row diagrams). Since the space $A_{n,[n/2]}$ is decomposed into the direct sum (7) of subspaces $H_{n,k}$ isomorphic to $A^0_{n,k}$, it suffices to define i_n on each $A^0_{n,k}$, $k = 0, 1, \dots, [n/2]$. Denote the corresponding operator on $A^0_{n,k}$ by $i_{n,k}$.

According to the fact that in the two-row part of the Young graph, the diagram $\lambda_{n,k}$ is joined with the diagrams $\lambda_{n+1,k}$ and (provided that $k + 1 \leqslant \frac{n+1}{2}$) $\lambda_{n+1,k+1}$, the operator $i_{n,k}$ is the sum

$$i_{n,k} = \xi_{n,k} + \eta_{n,k}, \tag{8}$$

where $\xi_{n,k}$ embeds $A^0_{n,k}$ into $A^0_{n+1,k}$ and $\eta_{n,k}$ embeds $A^0_{n,k}$ into $A^0_{n+1,k+1}$.

As to $\xi_{n,k}$, it is just the identity embedding mentioned above. Obviously, $\|\xi_{n,k} f\| = \|f\|$.

The nontrivial part is the embedding $\eta_{n,k}$, which is given by the following formula: for $f = \sum_{I \in F_{n,k}} c_I x_I \in A^0_{n,k}$, let

$$\tilde{\eta}_{n,k} f = \sum_{i=1}^{n} x_i \sum_{I \in F_{n,k}, I \not\ni i} c_I x_I - (n - 2k) x_{n+1} f = \sum_{J \in F_{n+1,k+1}} c_J x_J, \tag{9}$$

[1] The Koopman representation of a dynamical system (X, μ, G) is the representation T in the space $L^2(X, \mu)$ where $T_g f(x) = f(g^{-1}(x))$ for $g \in G$ and $T_\phi f = \phi f$ for $\phi \in L^\infty(X, \mu)$. In this case, the equivalence relation is the orbit partition. We use the term "Koopman representation" for the case of a groupoid algebra.

where

$$c_J = \begin{cases} \sum_{I\in F_{n,k}, I\subset J} c_I & \text{if } J \not\ni n+1; \\ -(n-2k)c_{J\setminus\{n+1\}} & \text{if } J \ni n+1. \end{cases}$$

Lemma 1. *Formula* (9) *defines an embedding from $A^0_{n,k}$ into $A^0_{n+1,k+1}$, and $\|\tilde\eta_{n,k} f\|^2 = c(n,k)\|f\|^2$, where $c(n,k) = (n-2k)(n-2k+1)$.*

Now let $\eta_{n,k} = \frac{1}{\sqrt{c(n,k)}}\tilde\eta_{n,k}$. Recall that i_n is the operator on $A_{n,[n/2]}$ whose restrictions to the subspaces $H_{n,k}$ isomorphic to $A^0_{n,k}$ are given by (8). The following proposition follows from construction and Lemma 1 by simple calculations.

Proposition 2. *The operator i_n determines an embedding from $A_{n,[n/2]}$ into $A_{n+1,[n+1/2]}$ which commutes with the action of the group $\mathfrak{S}_n$ and preserves the α-norms*: $\|i_n f\|_{\alpha,n+1} = \|f\|_{\alpha,n}$.

Denote by $\mathcal{A}_\alpha$ the inductive limit of the Hilbert spaces $(A_{n,[n/2]}, \|\cdot\|_\alpha)$ with respect to i_n and by ρ_α the corresponding limit unitary representation of $\mathfrak{S}_\infty$ in $\mathcal{A}_\alpha$.

Theorem 3. *The inductive limit ρ_α of the tensor representations of $\mathfrak{S}_n$ in the Hilbert spaces $(A_{n,[n/2]}, \|\cdot\|_{\alpha,n})$ with respect to the embeddings i_n is unitarily equivalent to the concomitant representation associated with the two-row central Thoma measure M^α.*

Proof. Follows from definitions and constructions by routine calculations using the representation theory of the symmetric groups. □

The concomitant representation associated with an ergodic central measure is irreducible. By definition, it agrees with the groupoid structure of the group algebra of $\mathfrak{S}_\infty$. Our goal is to describe it in "dynamical terms," i.e., to obtain for it an analog of the dynamical model for the tableau representation. Since the space $A_{n,[n/2]}$ can be identified with a subspace of functions on the space $\{0,1\}^n$ of finite 0–1 sequences (see the description of the first inductive limit above), Theorem 3 allows one to obtain a realization of the concomitant representation in terms of the Bernoulli scheme. However, the embeddings i_n are rather complicated, so that the identification of the limit space with some kind of "pseudo-functions" on the Bernoulli scheme is nontrivial. We will consider this problem elsewhere.

7. Appendix: on nonfree group actions

A specific feature of the dynamical model of the factor representations of $\mathfrak{S}_\infty$ is that it essentially exploits the "nonfree character" of the action of $\mathfrak{S}_\infty$ on $\mathcal{X}$, owing to which the cross product construction degenerates and the factor generated by the commutative algebra of functions on $\mathcal{X}$ and the group action operators coincides with the algebra generated only by the group action operators (provided that the measure m_α is not degenerate, i.e., the sequence α consists of pairwise distinct elements).

Apparently, the factors corresponding to nonfree actions are not sufficiently studied. Such actions have the following useful metric invariant, which vanishes in the case of free actions.

Definition 2. Assume that a countable group G acts on a measure space (X, ν) by measure-preserving transformations. Consider the mapping $x \mapsto G_x$ that associates with a point $x \in X$ its stabilizer regarded as an element of the space $\mathrm{Gr}(G)$ of all subgroups of the group G (with the ordinary Borel structure). Let $\bar{\nu}$ be the image of the measure ν under this mapping. We call it the *degree of nonfreedom of the action of* G.

The following proposition is obvious.

Proposition 3. *The degree of nonfreedom is a metric invariant of the action. In other words, if two actions of a countable group G are isomorphic, then the corresponding degrees of nonfreedom coincide.*

Example. For an action of the group $\mathbb{Z}$ on (X, μ), the degree of nonfreedom is a measure on the set of subgroups of $\mathbb{Z}$, and the measure of the subgroup $n\mathbb{Z}$ is equal to the μ-measure of the periodic orbits of pure period n. Thus if the action is periodic, then the degree of nonfreedom is its complete invariant.

In our case, this invariant allows us to distinguish the metric types of the actions of the infinite symmetric group $\mathfrak{S}_\infty$ on the space $\mathcal{X}$ with Bernoulli measures. For simplicity, consider the case of two states, i.e., consider measures $\alpha = (p, 1-p)$ and $\alpha' = (p', 1-p')$ with two nonzero components and the corresponding Bernoulli measures m_α and $m_{\alpha'}$ (we use the notation introduced in Section 4).

Proposition 4. *The actions of the infinite symmetric group $\mathfrak{S}_\infty$ on the space $\mathcal{X}$ with Bernoulli measures m_α and $m_{\alpha'}$ are metrically isomorphic only if the (unordered) pairs $\alpha = (p, 1-p)$ and $\alpha' = (p', 1-p')$ coincide.*

Proof. If $\alpha \neq \alpha'$, then the corresponding degrees of nonfreedom on the space of subgroups of $\mathfrak{S}_\infty$ are distinct. Indeed, consider the set of subgroups of the group $\mathfrak{S}_\infty$ in which there exists an element that sends the first coordinate to the second one:

$$\big\{H \in \mathrm{Gr}(\mathfrak{S}_\infty)\colon\ \exists h \in H\colon\ h(1) = 2\big\}.$$

Obviously, the degree of nonfreedom of this set for the action of $\mathfrak{S}_\infty$ on $(\mathcal{X}, m_\alpha)$ equals $m_\alpha(\{x\colon\ x_1 = x_2\}) = p^2 + (1-p)^2$. But these values do not coincide for $\alpha \neq \alpha'$. $\square$

In this case, the coincidence of the degrees of nonfreedom is also a sufficient condition for the metric isomorphism: if the (unordered) pairs $(p, 1-p)$ and $(p', 1-p')$ coincide, then there is an isomorphism, which is given either by the identity transformation (if $p = p'$), or by the permutation of states (if $p = 1 - p'$).

Acknowledgments

The authors are grateful to the anonymous referee for valuable comments.

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